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## SOME CONGRUENCES FOR THE SECOND-ORDER CATALAN NUMBERS

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ABSTRACT. Let  $p$  be any odd prime. We mainly show that

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv 0 \pmod{p}$$

and

$$\sum_{k=1}^{p-1} 2^{k-1} C_k^{(2)} \equiv (-1)^{(p-1)/2} - 1 \pmod{p},$$

where  $C_k^{(2)} = \binom{3k}{k} / (2k+1)$  is the  $k$ th Catalan number of order 2.

### 1. INTRODUCTION

The well-known Catalan numbers are those integers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} \quad (n = 0, 1, 2, \dots).$$

(As usual we regard  $\binom{x}{-k}$  as 0 for  $k = 1, 2, \dots$ .) There are many combinatorial interpretations for these important numbers (see, e.g., [St, pp. 219-229]). With the help of a sophisticated binomial identity, H. Pan and Z. W. Sun [PS] obtained some congruences on sums of Catalan numbers; in particular, by [PS, (1.16) and (1.8)], for any prime  $p > 3$  we have

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3\left(\frac{p}{3}\right) - 1}{2} \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{C_k}{k} \equiv \frac{3}{2} \left(1 - \left(\frac{p}{3}\right)\right) \pmod{p}, \quad (1.0)$$

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where the Legendre symbol  $\left(\frac{a}{3}\right) \in \{0, \pm 1\}$  satisfies the congruence  $a \equiv \left(\frac{a}{3}\right) \pmod{3}$ . Recently Z. W. Sun and R. Tauraso [ST1, ST2] obtained some further congruences concerning sums involving Catalan numbers.

For  $m, n \in \mathbb{N} = \{0, 1, 2, \dots\}$ , we define

$$C_n^{(m)} = \frac{1}{mn+1} \binom{mn+n}{n} = \binom{mn+n}{n} - m \binom{mn+n}{n-1}$$

and call it the  $n$ th Catalan number of order  $m$ . Clearly

$$C_n^{(1)} = C_n \quad \text{and} \quad C_n^{(2)} = \frac{1}{2n+1} \binom{3n}{n}.$$

In contrast with (1.0), we have the following result involving the second-order Catalan numbers.

**Theorem 1.1.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=1}^{p-1} 2^k C_k^{(2)} \equiv 2 \left( (-1)^{(p-1)/2} - 1 \right) \pmod{p} \quad (1.1)$$

and

$$\sum_{k=1}^{p-1} \frac{2^k C_k^{(2)}}{k} \equiv 4 \left( 1 - (-1)^{(p-1)/2} \right) \pmod{p}. \quad (1.2)$$

Actually Theorem 1.1 follows from our next two theorems.

**Theorem 1.2.** *Let  $p > 5$  be a prime. Then*

$$\sum_{k=0}^{p-1} 2^k \binom{3k}{k} \equiv \frac{6(-1)^{(p-1)/2} - 1}{5} \pmod{p}, \quad (1.3)$$

$$\sum_{k=0}^{p-1} 2^k \binom{3k+1}{k} \equiv \frac{4(-1)^{(p-1)/2} + 1}{5} \pmod{p}, \quad (1.4)$$

**Theorem 1.3.** *For any prime  $p$  we have*

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} \equiv 0 \pmod{p}. \quad (1.5)$$

For any odd prime  $p$  we can also prove the following congruences:

$$\begin{aligned} 5 \sum_{k=1}^{p-1} 2^k \binom{3k+2}{k} &\equiv (-1)^{(p-1)/2} - 1 \pmod{p}, \\ \sum_{k=1}^{p-1} \frac{2^{k-1}}{k} \binom{3k+1}{k} &\equiv (-1)^{(p-1)/2} - 1 \pmod{p}, \\ \sum_{k=1}^{p-1} \frac{2^{k-1}}{k} \binom{3k+2}{k} &\equiv \frac{3}{2} \left( (-1)^{(p-1)/2} - 1 \right) \pmod{p}. \end{aligned}$$

We omit their proofs which are similar to those of Theorems 1.2-1.3.

With the help of Theorems 1.2 and 1.3, we can easily deduce Theorem 1.1.

*Proof of Theorem 1.1 via Theorems 1.2 and 1.3.* Clearly (1.1) and (1.2) hold for  $p = 3, 5$ . Assume  $p > 5$ . By (1.3) and (1.4),

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{2^k}{2k+1} \binom{3k}{k} &= 3 \sum_{k=0}^{p-1} 2^k \binom{3k}{k} - 2 \sum_{k=0}^{p-1} 2^k \binom{3k+1}{k} \\ &\equiv 2(-1)^{(p-1)/2} - 1 \pmod{p}. \end{aligned}$$

This proves (1.1). For (1.2) it suffices to note that

$$\sum_{k=1}^{p-1} \frac{2^k}{k(2k+1)} \binom{3k}{k} = \sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} - 2 \sum_{k=1}^{p-1} \frac{2^k}{2k+1} \binom{3k}{k}.$$

This concludes the proof.  $\square$

We are going to provide two lemmas in the next section. Theorems 1.2 and 1.3 will be proved in Sections 3 and 4 respectively.

## 2. SOME LEMMAS

**Lemma 2.1.** *For  $m, n \in \mathbb{N}$  we have*

$$\begin{aligned} &2^n \sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{m-3k} \binom{3k-m+n}{k} \\ &= (-1)^m \sum_{j=0}^n \binom{n}{j} \sum_{k=0}^m (-2)^k \binom{n}{m-k} \binom{2j}{k}. \end{aligned} \tag{2.1}$$

*Proof.* Let  $P(x) = (2 + 2x - 4x^3)^n$ , and denote by  $[x^k]P(x)$  the coefficient of  $x^k$  in the expansion of  $P(x)$ . Then

$$\begin{aligned} 2^{-n} [x^m]P(x) &= [x^m]((1+x) - 2x^3)^n \\ &= \sum_{k=0}^{\lfloor m/3 \rfloor} \binom{n}{k} (-2)^k [x^{m-3k}](1+x)^{n-k} \\ &= \sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{k} \binom{n-k}{m-3k} \\ &= \sum_{k=0}^{\lfloor m/3 \rfloor} (-2)^k \binom{n}{m-3k} \binom{3k-m+n}{k}. \end{aligned}$$

Since

$$P(x) = (1-x)^n((2x+1)^2+1)^n = \sum_{j=0}^n \binom{n}{j} (1-x)^n (2x+1)^{2j},$$

we also have

$$[x^m]P(x) = \sum_{j=0}^n \binom{n}{j} \sum_{k=0}^m 2^k \binom{2j}{k} (-1)^{m-k} \binom{n}{m-k}.$$

Therefore (2.1) is valid.  $\square$

For any prime  $p$ , if  $n, k \in \mathbb{N}$  and  $s, t \in \{0, 1, \dots, p-1\}$  then we have the following well-known Lucas congruence (cf. [Gr] or [HS]):  $\binom{pn+s}{pk+t} \equiv \binom{n}{k} \binom{s}{t} \pmod{p}$ . This will be used in the proof of the following lemma.

**Lemma 2.2.** *Let  $p > 5$  be a prime. Then we have*

$$\sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{t} \equiv \frac{3(-1)^{(p-1)/2} + 2}{5} \pmod{p} \quad (2.2)$$

and

$$\sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{p+t} \equiv \frac{3}{10} \left(1 - (-1)^{(p-1)/2}\right) \pmod{p}. \quad (2.3)$$

*Proof.* Observe that

$$\begin{aligned} & \sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{t} \\ &= \sum_{s=0}^{(p-1)/2} (-1)^s \sum_{t=0}^{2s} 2^t \binom{2s}{t} + \sum_{s=(p+1)/2}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{t} \\ &= \sum_{s=0}^{(p-1)/2} (-1)^s 3^{2s} + \sum_{s=(p+1)/2}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{t} \\ &= \sum_{s=0}^{(p-1)/2} (-1)^s 3^{2s} + \sum_{s=(p+1)/2}^{p-1} (-1)^s \left( \sum_{t=0}^{2s} 2^t \binom{2s}{t} - \sum_{t=p}^{2s} 2^t \binom{2s}{t} \right) \\ &= \sum_{s=0}^{p-1} (-1)^s 3^{2s} - \sum_{s=(p+1)/2}^{p-1} (-1)^s \sum_{t=p}^{2s} 2^t \binom{2s}{t} \\ &= \sum_{s=0}^{p-1} (-9)^s - \sum_{s=(p+1)/2}^{p-1} (-1)^s \sum_{r=0}^{2s-p} 2^{p+r} \binom{2s}{r+p}. \end{aligned}$$

For  $s = (p+1)/2, \dots, p-1$ , by Lucas' congruence we have

$$\sum_{r=0}^{2s-p} 2^r \binom{p+(2s-p)}{p+r} \equiv \sum_{r=0}^{2s-p} 2^r \binom{2s-p}{r} = 3^{2s-p} \pmod{p}.$$

Thus, with the help of Fermat's little theorem, we get

$$\begin{aligned} \sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{t} &\equiv \frac{1 - (-9)^p}{10} - \sum_{s=(p+1)/2}^{p-1} (-1)^s \frac{2}{3} \cdot 9^s \\ &\equiv 1 - \frac{2}{3} (-9)^{\frac{p+1}{2}} \frac{(1 - (-9)^{(p-1)/2})}{10} \\ &\equiv \frac{3(-1)^{(p-1)/2} + 2}{5} \pmod{p}. \end{aligned}$$

This proves (2.2).

In view of Lucas' congruence and Fermat's little theorem, we also have

$$\begin{aligned} &\sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{p+t} \\ &\equiv \sum_{s=(p+1)/2}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s-p}{t} = \sum_{s=(p+1)/2}^{p-1} (-1)^s 3^{2s-p} \\ &= 3^{-p} (-9)^{(p+1)/2} \frac{1 - (-9)^{(p-1)/2}}{10} = (-1)^{(p+1)/2} \frac{3}{10} \left(1 + (-1)^{(p+1)/2} 3^{p-1}\right) \\ &\equiv \frac{3}{10} \left(1 - (-1)^{(p-1)/2}\right) \pmod{p}. \end{aligned}$$

So (2.3) is also valid. We are done.  $\square$

### 3. PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we first present an auxiliary result.

**Theorem 3.1.** *Let  $p > 5$  be a prime, and let  $d, \delta \in \{0, 1\}$ . Then*

$$\begin{aligned} &\frac{(-1)^{d+\delta}}{2^\delta} \sum_{\delta p - d \leq 3k \leq \delta p + p - 1 - d} 2^k \binom{3k+d}{k} \\ &\equiv \frac{4-\delta}{10} + \frac{(3\delta-2)(5d-3)}{10} (-1)^{(p-1)/2} \pmod{p}. \end{aligned} \tag{3.1}$$

*Proof.* Applying (2.1) with  $n = p-1$  and  $m = \delta p + p - 1 - d$ , we get

$$\begin{aligned} &2^{p-1} \sum_{k=0}^{\lfloor (\delta p + p - 1 - d)/3 \rfloor} (-2)^k \binom{p-1}{\delta p + p - 1 - d - 3k} \binom{3k+d-\delta p}{k} \\ &= (-1)^{\delta p + p - 1 - d} \sum_{j=0}^{p-1} \binom{p-1}{j} \sum_{k=0}^{\delta p + p - 1 - d} (-2)^k \binom{p-1}{\delta p + p - 1 - d - k} \binom{2j}{k}. \end{aligned}$$

Observe that

$$\begin{aligned}
& \sum_{k=0}^{\lfloor (\delta p + p - 1 - d)/3 \rfloor} (-2)^k \binom{p-1}{\delta p + p - 1 - d - 3k} \binom{3k + d - \delta p}{k} \\
&= \sum_{\delta p - d \leq 3k \leq \delta p + p - 1 - d} (-2)^k \binom{p-1}{p + \delta p - 1 - d - 3k} \binom{3k + d - \delta p}{k} \\
&\equiv \sum_{\delta p - d \leq 3k \leq \delta p + p - 1 - d} (-2)^k (-1)^{\delta p + p - 1 - d - 3k} \binom{3k + d}{k} \\
&\equiv (-1)^{d+\delta} \sum_{\delta p - d \leq 3k \leq \delta p + p - 1 - d} 2^k \binom{3k + d}{k} \pmod{p}
\end{aligned}$$

and

$$\begin{aligned}
& (-1)^{\delta p + p - 1 - d} \sum_{j=0}^{p-1} \binom{p-1}{j} \sum_{k=0}^{\delta p + p - 1 - d} (-2)^k \binom{p-1}{\delta p + p - 1 - d - k} \binom{2j}{k} \\
&\equiv \sum_{j=0}^{p-1} (-1)^j \sum_{\delta p - d \leq k < \delta p + p - d} 2^k \binom{2j}{k} = \sum_{j=0}^{p-1} (-1)^j \sum_{t=0}^{p-1} 2^{\delta p - d + t} \binom{2j}{\delta p - d + t} \\
&\equiv 2^{\delta - d} \sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{\delta p - d + t} \pmod{p}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{\delta p - d \leq 3k \leq \delta p + p - 1 - d} 2^k \binom{3k + d}{k} \\
&\equiv (-2)^{\delta - d} \sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{\delta p - d + t} \pmod{p}.
\end{aligned}$$

Recall that  $d \in \{0, 1\}$ . We have

$$\begin{aligned}
& \sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{\delta p - d + t} \\
&= \sum_{s=0}^{p-1} (-1)^s \sum_{t=-d}^{p-1-d} 2^{d+t} \binom{2s}{\delta p + t} \\
&= \sum_{s=0}^{p-1} (-1)^s \left( \sum_{t=0}^{p-1} 2^{d+t} \binom{2s}{\delta p + t} + d \left( \binom{2s}{\delta p - 1} - 2^p \binom{2s}{\delta p + p - 1} \right) \right) \\
&= 2^d \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} (-1)^s 2^t \binom{2s}{\delta p + t} + d \sum_{s=0}^{p-1} (-1)^s \left( \binom{2s}{\delta p - 1} - 2^p \binom{2s}{\delta p + p - 1} \right)
\end{aligned}$$

and hence

$$\begin{aligned}
& (-1)^{d+\delta} \sum_{\delta p-d \leq 3k \leq \delta p+p-1-d} 2^k \binom{3k+d}{k} - 2^\delta \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} (-1)^s 2^t \binom{2s}{\delta p+t} \\
& \equiv d 2^{\delta-d} \sum_{s=0}^{p-1} (-1)^s \left( \binom{2s}{\delta p-1} - 2 \binom{2s}{\delta p+p-1} \right) \\
& \equiv d 2^{\delta-1} (3\delta-2) \sum_{s=0}^{p-1} (-1)^s \binom{2s}{p-1} \equiv d(3\delta-2) 2^{\delta-1} (-1)^{(p-1)/2} \pmod{p}.
\end{aligned}$$

Since

$$\sum_{s=0}^{p-1} (-1)^s \sum_{t=0}^{p-1} 2^t \binom{2s}{\delta p+t} \equiv \frac{4-\delta}{10} + \frac{3}{10} (2-3\delta) (-1)^{(p-1)/2} \pmod{p}$$

by Lemma 2.2, we finally get

$$\begin{aligned}
& \frac{(-1)^{d+\delta}}{2^\delta} \sum_{\delta p-d \leq 3k \leq \delta p+p-1-d} 2^k \binom{3k+d}{k} \\
& \equiv \frac{4-\delta}{10} + \frac{3}{10} (2-3\delta) (-1)^{(p-1)/2} + \frac{d}{2} (3\delta-2) (-1)^{(p-1)/2} \\
& \equiv \frac{4-\delta}{10} + \frac{(3\delta-2)(5d-3)}{10} (-1)^{(p-1)/2} \pmod{p}.
\end{aligned}$$

This proves (3.1).  $\square$

*Proof of Theorem 1.2.* Let  $d \in \{0, 1\}$ . If  $(2p-d)/3 \leq k \leq p-1$ , then  $2k+d+1 \leq 2k+2 \leq 2p \leq 3k+d$  and hence

$$\binom{3k+d}{k} = \frac{(3k+d) \cdots (2k+d+1)}{k!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{2p-d \leq 3k \leq 3p-3} 2^k \binom{3k+d}{k} \equiv 0 \pmod{p}.$$

With the help of Theorem 3.1, we have

$$\begin{aligned}
\sum_{k=0}^{p-1} 2^k \binom{3k+d}{k} & \equiv \sum_{-d \leq 3k \leq 2p-1-d} 2^k \binom{3k+d}{k} \\
& \equiv \sum_{\delta=0}^1 \sum_{\delta p-d \leq 3k \leq \delta p+p-1-d} 2^k \binom{3k+d}{k} \\
& \equiv \sum_{\delta=0}^1 (-1)^d (-2)^\delta \left( \frac{4-\delta}{10} + \frac{(3\delta-2)(5d-3)}{10} (-1)^{(p-1)/2} \right) \\
& \equiv \frac{(-1)^{d-1}}{5} \left( 1 + (10d-6) (-1)^{(p-1)/2} \right) \pmod{p}.
\end{aligned}$$

This yields (1.3) and (1.4). We are done.  $\square$

#### 4. PROOF OF THEOREM 1.3

*Proof of Theorem 1.3.* Obviously (1.5) holds for  $p = 2, 3$ . Below we assume  $p > 3$ .

Let  $\delta \in \{0, 1\}$ . Applying (2.1) with  $m = p + \delta p$  and  $n = p$  we get

$$\begin{aligned} & 2^p \sum_{k=0}^p (-2)^k \binom{p}{p + \delta p - 3k} \binom{3k - \delta p}{k} \\ &= (-1)^{\delta+1} \sum_{j=0}^p \binom{p}{j} \sum_{k=0}^{p+\delta p} (-2)^k \binom{p}{p + \delta p - k} \binom{2j}{k}. \end{aligned} \tag{4.1}$$

Observe that

$$\begin{aligned} & \sum_{k=0}^p (-2)^k \binom{p}{p + \delta p - 3k} \binom{3k - \delta p}{k} \\ &= \sum_{\delta p \leq 3k \leq p + \delta p - 1} (-2)^k \binom{p}{3k - \delta p} \binom{3k - \delta p}{k} \\ &= 1 - \delta + \sum_{\delta p < 3k < p + \delta p} (-2)^k \binom{p}{3k - \delta p} \binom{3k - \delta p}{k}. \end{aligned}$$

For  $j = 1, \dots, p-1$  clearly

$$\binom{p}{j} = \frac{p}{j} \binom{p-1}{j-1} \equiv p \frac{(-1)^{j-1}}{j} \pmod{p^2}.$$

Thus

$$\begin{aligned} & \sum_{\delta p < 3k < p + \delta p} (-2)^k \binom{p}{3k - \delta p} \binom{3k - \delta p}{k} \\ &\equiv \sum_{\delta p < 3k < p + \delta p} (-2)^k p \frac{(-1)^{3k - \delta p - 1}}{3k - \delta p} \binom{3k - \delta p}{k} \\ &\equiv (-1)^{\delta+1} \sum_{\delta p < 3k < p + \delta p} (-2)^k p \frac{(-1)^k}{3k} \binom{(3k - \delta p) + \delta p}{k} \\ &\quad \text{(by Lucas' congruence)} \\ &\equiv (-1)^{\delta+1} \frac{p}{3} \sum_{\delta p < 3k < p + \delta p} \frac{2^k}{k} \binom{3k}{k} \pmod{p^2}. \end{aligned}$$

Notice that

$$\begin{aligned}
& \sum_{j=0}^p \binom{p}{j} \sum_{k=0}^{p+\delta p} (-2)^k \binom{p}{p+\delta p-k} \binom{2j}{k} \\
&= \sum_{\delta p \leq 2j \leq 2p} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k} \\
&= \sum_{\delta p < 2j < 2p} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k} \\
&\quad + \sum_{2j \in \{\delta p, 2p\}} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k}.
\end{aligned}$$

Clearly

$$\begin{aligned}
& \sum_{\delta p < 2j < 2p} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k} \\
&\equiv \sum_{\delta p < 2j < 2p} \binom{p}{j} \left( (-2)^{\delta p} \binom{p}{0} \binom{2j}{\delta p} + (-2)^{p+\delta p} \binom{p}{p} \binom{2j}{p+\delta p} \right) \\
&\equiv \sum_{\delta p < 2j < 2p} \binom{p}{j} (-2)^{\delta p} \binom{2j-\delta p}{0} \\
&\quad + (1-\delta) \sum_{p < 2j < 2p} \binom{p}{j} (-2)^{p+\delta p} \binom{2j-p}{p-p} \text{ (by Lucas' congruence)} \\
&\equiv (-2)^{\delta} 2^{1-\delta} (2^{p-1} - 1) + (1-\delta) (-2)^{1+\delta} (2^{p-1} - 1) \\
&\equiv (-1)^{\delta} \delta (2^p - 2) = -\delta (2^p - 2) \pmod{p^2}.
\end{aligned}$$

(Note that  $\delta \in \{0, 1\}$  and  $2 \sum_{p/2 < j < p} \binom{p}{j} = \sum_{j=1}^{p-1} \binom{p}{j} = 2^p - 2$ .) Also,

$$\sum_{2j=\delta p} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k} = (1-\delta) \sum_{k=0}^p (-2)^k \binom{p}{k} \binom{0}{k} = 1 - \delta$$

and

$$\begin{aligned}
& \sum_{2j=2p} \binom{p}{j} \sum_{k=\delta p}^{p+\delta p} (-2)^k \binom{p}{k-\delta p} \binom{2j}{k} \\
&\equiv \sum_{k \in \{\delta p, p+\delta p\}} (-2)^k \binom{p}{k-\delta p} \binom{2p}{k} \\
&\equiv (-2)^{\delta p} \binom{2}{\delta} + (-2)^{p+\delta p} \binom{2}{1+\delta} = 4^{\delta p} - 2^{p+1} \pmod{p^2}.
\end{aligned}$$

(Recall that  $\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$  by the Wolstenholme congruence (cf. [Gr] or [HT]).)

Combining the above with (4.1), we have

$$\begin{aligned} & 2^p \left( 1 - \delta + (-1)^{\delta+1} \frac{p}{3} \sum_{\delta p < 3k < p+\delta p} \frac{2^k}{k} \binom{3k}{k} \right) \\ & \equiv (-1)^{\delta+1} (\delta(2-2^p) + 1 - \delta + 4^{\delta p} - 2^{p+1}) \pmod{p^2}. \end{aligned}$$

Setting  $\delta = 0$  and  $\delta = 1$  respectively, we obtain

$$2^p - 2^p \frac{p}{3} \sum_{0 < 3k < p} \frac{2^k}{k} \binom{3k}{k} \equiv 2^{p+1} - 2 \pmod{p^2}$$

and

$$2^p \frac{p}{3} \sum_{p < 3k < 2p} \frac{2^k}{k} \binom{3k}{k} \equiv 2 - 2^p + 4^p - 2^{p+1} \pmod{p^2}.$$

It follows that

$$\frac{2}{3}p \sum_{0 < 3k < 2p} \frac{2^k}{k} \binom{3k}{k} \equiv 4^p - 4 \cdot 2^p + 4 = (2^p - 2)^2 \equiv 0 \pmod{p^2}.$$

If  $2p \leq 3k < 3p$ , then

$$\binom{3k}{k} = \frac{3k \cdots (2k+1)}{k!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \binom{3k}{k} = \sum_{0 < 3k < 2p} \frac{2^k}{k} \binom{3k}{k} + \sum_{2p \leq 3k < 3p} \frac{2^k}{k} \binom{3k}{k} \equiv 0 \pmod{p}.$$

This completes the proof of Theorem 1.3.  $\square$

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